

FERMIONIC NOVIKOV ALGEBRAS ADMITTING INVARIANT NON-DEGENERATE SYMMETRIC BILINEAR FORMS ARE NOVIKOV ALGEBRAS

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ABSTRACT. This paper is to prove that a fermionic Novikov algebra equipped with an invariant non-degenerate symmetric bilinear form is a Novikov algebra.

1. INTRODUCTION

Gel'fand and Dikii gave a bosonic formal variational calculus in [5, 6] and Xu gave a fermionic formal variational calculus in [13]. Combining the bosonic theory of Gel'fand-Dikii and the fermionic theory, Xu gave in [14] a formal variational calculus of super-variables. Fermionic Novikov algebras are related to the Hamiltonian super-operator in terms of this theory. A fermionic Novikov algebra is a finite-dimensional vector space A over a field \mathbb{F} with a bilinear product $(x, y) \mapsto xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \quad (1.1)$$

$$(xy)z = -(xz)y \quad (1.2)$$

for any $x, y, z \in A$. It corresponds to the following Hamiltonian operator H of type 0 ([14]):

$$H_{\alpha, \beta}^0 = \sum_{\gamma \in I} (a_{\alpha, \beta}^{\gamma} \Phi_{\gamma}(2) + b_{\alpha, \beta}^{\gamma} \Phi_{\gamma} D), \quad a_{\alpha, \beta}^{\gamma}, b_{\alpha, \beta}^{\gamma} \in \mathbb{R}. \quad (1.3)$$

Fermionic Novikov algebras are a class of left-symmetric algebras which are defined by the identity (1.1). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones ([2, 12]). Novikov algebras are another class of left-symmetric algebras A satisfying

$$(xy)z = (xz)y, \quad \forall x, y, z \in A \quad (1.4)$$

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type ([1, 3, 4]) and Hamiltonian operators in the formal variational calculus ([5, 6, 7, 13, 15]).

The commutator of a left-symmetric algebra A

$$[x, y] = xy - yx \quad (1.5)$$

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defines a Lie algebra, which is called the underlying Lie algebra of A . A bilinear form $\langle \cdot, \cdot \rangle$ on a left-symmetric algebra A is invariant if

$$\langle R_x y, z \rangle = \langle y, R_x z \rangle \quad (1.6)$$

for any $x, y, z \in A$.

Zelmanov ([16]) classifies real Novikov algebras with invariant positive definite symmetric bilinear forms. In [8], Guediri gives the classification for the Lorentzian case. This paper is to study real fermionic Novikov algebras admitting invariant non-degenerate symmetric bilinear forms. The main result is the following theorem.

Theorem 1.1. *Any finite dimensional real fermionic Novikov algebra admitting an invariant non-degenerate symmetric bilinear form is a Novikov algebra.*

In order to prove Theorem 1.1, we describe the structure of these fermionic Novikov algebras. But we only give part of the classification since the complete classification is very complicated.

2. THE PROOF OF THEOREM 1.1

Let A be a fermionic Novikov algebra and let L_x and R_x denote the left and right multiplication operator by the element $x \in A$ respectively, i.e.,

$$L_x(y) = xy, \quad R_x(y) = yx$$

for any $y \in A$. By the equation (1.2), we have

$$R_x R_y = -R_y R_x, \quad \forall x, y \in A.$$

In particular, $R_x^2 = 0$ for any $x \in A$.

Definition 2.1. A non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on V is of type $(n - p, p)$ if there is a basis $\{e_1, \dots, e_n\}$ of V such that $\langle e_i, e_i \rangle = -1$ for $1 \leq i \leq p$, $\langle e_i, e_i \rangle = 1$ for $p + 1 \leq i \leq n$, and $\langle e_i, e_j \rangle = 0$ for otherwise. The bilinear form is positive definite if $p = 0$; Lorentzian if $p = 1$.

A linear operator σ of $(V, \langle \cdot, \cdot \rangle)$ is self-adjoint if

$$\langle \sigma(x), y \rangle = \langle x, \sigma(y) \rangle, \quad \forall x, y \in V.$$

Lemma 2.2 ([10], pp. 260-261). *A linear operator σ on $V = \mathbb{R}^n$ is self-adjoint if and only if V can be expressed as a direct sum of V_k that are mutually orthogonal (hence non-degenerate),*

σ -invariant, and each $\sigma|_{V_k}$ has a $r \times r$ matrix form either

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 1 & \lambda & \cdots & \vdots \\ \vdots & \ddots & \lambda & 0 \\ 0 & \cdots & 1 & \lambda \end{pmatrix}$$

relative to a basis $\alpha_1, \dots, \alpha_r$ ($r \geq 1$) with all scalar products zero except $\langle \alpha_i, \alpha_j \rangle = \pm 1$ if $i + j = r + 1$, or

$$\begin{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & & & \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & \\ 0 & & \ddots & \ddots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}$$

where $b \neq 0$ relative to a basis $\beta_1, \alpha_1, \dots, \beta_m, \alpha_m$ with all scalar products zero except $\langle \beta_i, \beta_j \rangle = 1 = -\langle \alpha_i, \alpha_j \rangle$ if $i + j = m + 1$.

If A admits an invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of type $(n - p, p)$, then $-\langle \cdot, \cdot \rangle$ is an invariant non-degenerate symmetric bilinear form on A of type $(p, n - p)$. So we can assume $p \leq n - p$.

Lemma 2.3. *Let A be a fermionic Novikov algebra admitting an invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of type $(n - p, p)$, then $\dim \text{Im} R_x \leq p$ for any $x \in A$.*

Proof. Recall that $R_x^2 = 0$, it follows that $\text{Im} R_x \subseteq \text{Ker} R_x$. By the invariance of $\langle \cdot, \cdot \rangle$, we have $\langle R_x y, R_x z \rangle = \langle y, R_x^2 z \rangle = 0$ which yields $\langle \text{Im} R_x, \text{Im} R_x \rangle = 0$. Hence $\dim \text{Im} R_x \leq p$. \square

Let $x_0 \in A$ satisfy $\dim \text{Im} R_x \leq \dim \text{Im} R_{x_0}$ for any $x \in A$. By Lemma 2.3, $\dim \text{Im} R_{x_0} \leq p$. For convenience, let $\dim \text{Im} R_{x_0} = k$. By Lemma 2.2 and $R_{x_0}^2 = 0$, there exists a basis $\{e_1, \dots, e_n\}$ of A such that the operator R_{x_0} relative to the basis has the matrix of form

$$\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & 0 \\ & \ddots \\ & 0 & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}_{2k \times 2k} & 0_{2k \times (n-2k)} \\ 0_{(n-2k) \times 2k} & 0_{(n-2k) \times (n-2k)} \end{pmatrix},$$

where the matrix of the metric $\langle \cdot, \cdot \rangle$ with respect to $\{e_1, \dots, e_n\}$ is

$$\begin{pmatrix} C_{2k} & 0 & 0 \\ 0 & -I_{p-k} & 0 \\ 0 & 0 & I_{n-p-k} \end{pmatrix}.$$

Here $C_{2k} = \text{diag} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ and I_s denotes the $s \times s$ identity matrix. For any $x \in A$, the matrix of the operator R_x relative to the basis is

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix},$$

whose blocks are the same as those of the metric matrix under the basis $\{e_1, \dots, e_n\}$.

Firstly we can prove that

$$\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix} = 0_{(n-2k) \times (n-2k)}.$$

In fact, assume that there exists some nonzero entry of $\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix}$ which denoted by d . Consider the matrix form of the operator $R_x + lR_{x_0}$. With no confusions, we do not distinguish between the operator R_x and its matrix form in the following. For any $l \in \mathbb{R}$, by the choice of x_0 , we know that $r(R_x + lR_{x_0}) = r(R_{x+lx_0}) \leq k$. Taking 2nd, \dots , $2k$ -th rows, 1st, \dots , $(2k-1)$ -th columns, and the row and column containing the element d in the matrix of $R_x + lR_{x_0}$, we have the $(k+1) \times (k+1)$ matrix $\begin{pmatrix} B + lI_k & \alpha \\ \beta & d \end{pmatrix}$. Note that the determinant of $\begin{pmatrix} B + lI_k & \alpha \\ \beta & d \end{pmatrix}$, i.e.,

$$\begin{vmatrix} B + lI_k & \alpha \\ \beta & d \end{vmatrix},$$

is a polynomial of degree k in a single indeterminate l . So we can choose some $l' \in \mathbb{R}$ such that the determinant is nonzero. It follows that

$$r(R_x + l'R_{x_0}) = r(R_{x+l'x_0}) \geq k+1,$$

which is a contradiction.

Secondly, by $R_x R_{x_0} + R_{x_0} R_x = 0$, we have that $A_1 = (M_{ij})_{k \times k}$ where $M_{ij} = \begin{pmatrix} b_{ij} & 0 \\ d_{ij} & -b_{ij} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{2,1} & \cdots & \cdots & a_{2,p-k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ a_{2k,1} & \cdots & \cdots & a_{2k,p-k} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ c_{2,1} & \cdots & \cdots & c_{2,n-p-k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \\ c_{2k,1} & \cdots & \cdots & c_{2k,n-p-k} \end{pmatrix}.$$

Furthermore, since $\langle R_x y, z \rangle = \langle y, R_x z \rangle$, we obtain that

$$M_{ij} = \begin{pmatrix} b_{ij} & 0 \\ d_{ij} & -b_{ij} \end{pmatrix}, M_{ji} = \begin{pmatrix} -b_{ij} & 0 \\ d_{ij} & b_{ij} \end{pmatrix},$$

where $b_{ii} = 0$ for any $1 \leq i \leq k$, and

$$A_4 = - \begin{pmatrix} a_{2,1} & 0 & \cdots & a_{2k,1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2,p-k} & 0 & \cdots & a_{2k,p-k} & 0 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} c_{2,1} & 0 & \cdots & c_{2k,1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{2,n-p-k} & 0 & \cdots & c_{2k,n-p-k} & 0 \end{pmatrix}.$$

Since $R_x^2 = 0$, we have that $A_1^2 + A_2 A_4 + A_3 A_7 = 0_{2k \times 2k}$. Note that

$$0 = (A_1^2 + A_2 A_4 + A_3 A_7)_{i,i} = (A_1^2)_{i,i}.$$

It follows that $b_{ij} = 0$ for any i, j . Then

$$M_{ij} = M_{ji} = \begin{pmatrix} 0 & 0 \\ d_{ij} & 0 \end{pmatrix}.$$

Finally, we claim that A_2, A_3, A_4 and A_7 are zero matrices. Here we only prove $A_2 = 0_{2k \times (p-k)}$, similar for others. Assume that there exists some nonzero entry of A_2 which denoted by d . Consider the matrix of the operator $R_x + lR_{x_0}$. Similar to the proof of

$$\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix} = 0_{(n-2k) \times (n-2k)},$$

we consider the matrix

$$\begin{pmatrix} A'_1 + lI_k & \alpha^T \\ -\alpha & 0 \end{pmatrix},$$

where d is an entry in the vector α and $A'_1 = (d_{ij})_{k \times k}$ is a symmetric matrix. Thus there exists an orthogonal matrix P such that $P^T A'_1 P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$. Choose some $l > \{|\lambda_1|, \dots, |\lambda_k|\}$.

Then the matrix $A'_1 + lI_k$ is invertible. We have

$$\begin{aligned} & \left| \begin{pmatrix} A'_1 + lI_k & \alpha^T \\ -\alpha & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} P^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A'_1 + lI_k & \alpha^T \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \right| \\ & = \left| \begin{pmatrix} \lambda_1 + l & & 0 \\ & \ddots & \\ 0 & & \lambda_k + l \end{pmatrix} \begin{pmatrix} \beta^T \\ -\beta \\ 0 \end{pmatrix} \right| = (\prod_{i=1}^k (\lambda_i + l)) \sum_{i=1}^k \frac{1}{\lambda_i + l} b_i^2 \neq 0, \end{aligned}$$

where $\beta = \alpha P = (b_1, \dots, b_k)$ is a nonzero vector. It follows that

$$r(R_x + lR_{x_0}) = r(R_{x+lx_0}) \geq k + 1,$$

which is a contradiction. That is, $A_2 = 0_{2k \times (p-k)}$.

Up to now, we know that the matrix of R_x is

$$\begin{pmatrix} A_1 & 0_{2k \times (n-2k)} \\ 0_{(n-2k) \times 2k} & 0_{(n-2k) \times (n-2k)} \end{pmatrix},$$

where $A_1 = (M_{ij})_{k \times k}$, here $M_{ij} = M_{ji} = \begin{pmatrix} 0 & 0 \\ d_{ij}(x) & 0 \end{pmatrix}$. Hence $R_x R_y = 0$ for any $x, y \in A$, which implies Theorem 1.1.

3. THE STRUCTURE OF SUCH FERMIONIC NOVIKOV ALGEBRAS

Let A be an n -dimensional fermionic Novikov algebra with an invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of type $(n-p, p)$. By the above section, if $x_0 \in A$ satisfies

$$\dim \text{Im} R_x \leq \dim \text{Im} R_{x_0} = k \leq p$$

for any $x \in A$, then there exists a basis $\{e_1, \dots, e_n\}$ such that the matrix of R_x is

$$\begin{pmatrix} A_1 & 0_{2k \times (n-2k)} \\ 0_{(n-2k) \times 2k} & 0_{(n-2k) \times (n-2k)} \end{pmatrix},$$

where $A_1 = (M_{ij})_{k \times k}$, here $M_{ij} = M_{ji} = \begin{pmatrix} 0 & 0 \\ d_{ij}(x) & 0 \end{pmatrix}$. In particular, $d_{ii}(x_0) = 1$ for $i = 1, \dots, k$ and others zero. Clearly

Proposition 3.1. $\dim AA = \dim \text{Im} R_{x_0} = k$.

If $k = 0$, then $xy = 0$ for any $x, y \in A$.

If $k = 1$, then there exists a basis $\{e_1, \dots, e_n\}$ such that the matrix of R_x is

$$\begin{pmatrix} M & 0_{2 \times (n-2)} \\ 0_{(n-2) \times 2} & 0_{(n-2) \times (n-2)} \end{pmatrix},$$

where $M = \begin{pmatrix} 0 & 0 \\ d(x) & 0 \end{pmatrix}$. Clearly the matrices of L_{e_i} are zero matrices if $i \neq 1$. Thus

$$L_x L_y = L_y L_x, \quad \forall x, y \in A.$$

Together with $R_x R_y = 0$ for any $x, y \in A$, the matrices of R_{e_i} for $1 \leq i \leq n$ determine a fermionic Novikov algebra. Furthermore A is one of the following cases:

- (1) $k = 1$, and there exists a basis $\{e_1, \dots, e_n\}$ such that $e_1 e_1 = e_2$ and others zero.
- (2) $k = 1$, and there exists a basis $\{e_1, \dots, e_n\}$ such that $e_1 e_2 = e_2$ and others zero.
- (3) $k = 1$, and there exists a basis $\{e_1, \dots, e_n\}$ such that $e_1 e_3 = e_2$ and others zero.

In particular, the above discussion gives the classification of fermionic Novikov algebras admitting invariant Lorentzian symmetric bilinear forms which is obtained in [8].

If $k = 2$, then there exists a basis $\{e_1, \dots, e_n\}$ such that nonzero products are given by

$$e_1 e_i = \lambda_i e_2 + \mu_i e_4, \quad e_3 e_i = \mu_i e_2 + \gamma_i e_4.$$

For this case, A is a fermionic Novikov algebra if and only if $L_{e_1} L_{e_3} = L_{e_3} L_{e_1}$. But the complete classification is very complicated. It is similar for $k \geq 3$.

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